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A matrix approach to the study of wave propagation

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Abstract. Wave propagation in piecewise continuous stratified non-absorbing media is discussed. The general solution of the amplitude equation is expressed as a linear combination of two independent complex conjugate solutions. The transmission matrix relating the amplitudes at two points belongs to the group QU(2), in accord with energy conservation. An upper bound for the reflectance is calculated. In the high frequency limit, the reflection coefficient r corresponding to a discontinuity of the derivative of the refractive index is found to be inversely proportional to the frequency. The eigenfrequencies of the field are shown to be completely determined by the reflection and transmission coefficients of the medium.

1. Introduction

In a recent paper (Robnik 1979, hereafter cited as I) a matrix method has been used to study solutions of the one-dimensional amplitude equation

$$u'' - (\nu'/\nu)u' + k^2 u = 0, \tag{1}$$

where $k(x)$ is the x component of the wave vector, i.e. the component perpendicular to the strata. In the case of electromagnetic waves one must consider the two polarisations. If the electric field is normal to the plane of incidence, $u(x)$ denotes the complex amplitude of that field and $\nu' = 0$ (Landau and Lifshitz 1967); for formal reasons we shall use $\nu = 1$. In the other case u is the amplitude of the magnetic field and $\nu(x)$ equals the dielectric constant. The parameters ν and k are assumed to be real and positive, so that only non-absorbing stratified media are considered.

The matrix approach in I is based on a step-function approximation for the parameters ν and k, so that on each step u is expressed as a linear combination of $\exp(\mathrm{i}kx)$ and $\exp(-\mathrm{i}kx)$. But this is a rather special case which can be generalised very easily, as will be shown in subsequent sections. The generalised matrix formalism leads to some useful specific results and seems to be the proper starting point for a generalisation of the step-function approximation.

2. Transmission matrices and their group property

We shall analyse wave propagation in piecewise continuous media, such that the profile of each individual layer (x_j, x_{j+1}), $j = 1, 2, \ldots, n-1$, can be exactly described by the

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functions $\nu = \nu_j(x)$ and $k = k_j(x)$. (The free space on the left will be labeled by 0, and on the right from the medium by n.) We assume that there exists in each layer a pair (ψ_j, ψ_j^*) of exact, linearly independent and complex conjugate solutions, so that the purely imaginary Wronskian

$$W = \psi\, \partial_x\psi^* - \psi^*\, \partial_x\psi = -W^* \tag{2}$$

does not vanish. The ordering of the pair (ψ, ψ^*) is determined by the choice Im $W < 0$, so that $\mathrm{i}W = |W|$. The solution of equation (1) is then written as a linear combination

$$u(x) = |\nu/W|^{1/2}(A\psi(x) + B\psi^*(x)), \tag{3}$$

where for later convenience the constant ratio ν/W has been included in the definition. The constancy of ν/W follows from the Liouville theorem as applied to equation (1). The complex amplitudes A and B are constant in the given layer and are conveniently summarised in a two-dimensional complex amplitude vector $v = (A, B)$. By aid of the vector $f(x) = (\psi^*(x), \psi(x))$ equation (3) can be rewritten in the form of the usual inner product,

$$u = |\nu/W|^{1/2}\langle v, f\rangle. \tag{4}$$

Our aim is to determine how v transforms on transition from layer to layer. That is, we are looking for a relation between the amplitude vectors v_+ and v_- on the right and on the left of a discontinuity. The boundary conditions which require that $u(x)$ and $u'(x)/\nu$ are continuous can be expressed in matrix form as follows:

$$S_+v_+ = S_-v_-, \tag{5}$$

$$S = |\nu/W|^{1/2}\begin{pmatrix} \psi, & \psi^* \\ \nu^{-1}\,\partial_x\psi, & \nu^{-1}\,\partial_x\psi^* \end{pmatrix}. \tag{6}$$

Obviously $\det S = -\mathrm{i}$, so that S^{-1} exists. Thus

$$v_+ = Gv_-, \tag{7}$$

with the transmission matrix

$$G = S_+^{-1}S_-. \tag{8}$$

Since the boundary conditions (5, 6) guarantee that energy conservation is obeyed, an invariant proportional to the total energy flux in the x direction must exist. Repeating the argument given by Heading (1975) we multiply equation (1) by u^* and integrate the imaginary part of the resulting expression. Thus

$$J = \mathrm{Im}(u'u^*/\nu) = \pm\tfrac{1}{2}(|A|^2 - |B|^2) = \text{constant}, \tag{9}$$

where the solution (3) has been inserted. In our case the positive sign, which corresponds to Im $W < 0$, must be taken. According to this convention A is always associated with the forward energy flux, while B refers to the backward flux. For instance, if $\nu' = 0$ and $k = $ constant, we have $\psi = \exp(\mathrm{i}kx)$ and $W = -2\mathrm{i}k$, i.e. Im $W < 0$. If the time factor is given by $\exp(-\mathrm{i}\omega t)$, then A refers to the incident wave.

The existence of the invariant J from (9) implies certain symmetry properties of the transmission matrix (8). In particular, since $\det G = 1$ each transmission matrix belongs to the group QU(2) (Vilenkin 1965), and can be written in terms of three real parameters

$$G=\begin{pmatrix}\exp[\mathrm{i}(\phi+\chi)]\cosh\tau, & \exp[\mathrm{i}(\phi-\chi)]\sinh\tau\\ \exp[-\mathrm{i}(\phi-\chi)]\sinh\tau, & \exp[-\mathrm{i}(\phi+\chi)]\cosh\tau\end{pmatrix} \tag{10}$$

where $\tau\in(-\infty,\infty)$, while ϕ and χ are confined to the interval $[0, 2\pi]$ if uniqueness of the representation (10) is required. We notice that

$$\begin{aligned}G_{11}=G_{22}^{*}&=\mathrm{i}|\nu_+\nu_-/W_+W_-|^{1/2}(\nu_+^{-1}\psi_-\,\partial_x\psi_+^{*}-\nu_-^{-1}\psi_+^{*}\,\partial_x\psi_-),\\ G_{21}=G_{12}^{*}&=-\mathrm{i}|\nu_+\nu_-/W_+W_-|^{1/2}(\nu_+^{-1}\psi_-\,\partial_x\psi_+-\nu_-^{-1}\psi_+\,\partial_x\psi_-),\end{aligned} \tag{11}$$

where ψ_+ and ψ_- refer to the right and left sides of the discontinuity respectively.

For n discontinuities at the points $x_1, x_2, \ldots, x_n$ the transmission matrix G corresponding to the whole stratified structure will be equal to the product

$$G=G_nG_{n-1}\ldots G_2G_1=\prod_{j=1}^{j=n}G_j, \tag{12}$$

so that $v_n=Gv_0$. Since the resulting transmission matrix G again belongs to QU(2), it too can be expressed in the form (10).

If for $x>x_n$ there is only a transmitted wave, i.e. $v_n=(A_n, 0)$, the identity

$$G_{21}A_0+G_{22}B_0=0$$

must be satisfied, whence an expression for the reflection coefficient r of the multilayer structure follows:

$$r=B_0\psi_0^{*}(x_1)/A_0\psi_0(x_1)=-G_{21}\psi_0^{*}(x_1)/G_{22}\psi_0(x_1). \tag{13}$$

With the aid of expression (10) this can be rewritten in parameter form

$$r=-(\exp(2\mathrm{i}\chi)\tanh\tau)(\psi_0^{*}/\psi_0), \tag{14}$$

where the argument of the ψ's has been omitted. The reflectance equals

$$|r|^2=\tanh^2\tau.$$

The reflection coefficient r as defined in equation (13) is what is needed for the physics. However, for mathematical manipulation the ratios $R=B_0/A_0$ and $T=A_n/A_0$ may be more suitable. For the incidence from the right, i.e. $v_n=(A_n', B_n')$ and $v_0=(0, B_0')$, we define correspondingly $R'=A_n'/B_n'$ and $T'=B_0'/B_n'$. Then the simple algebraic equations $v_n=Gv_0$ lead to the result

$$T=T'=1/G_{22}, \qquad R'/T'=G_{12}/G_{22}=-R^{*}/T^{*}. \tag{15}$$

These symmetries have already been proven by Heading (1978). One should notice that the symmetry properties of the transmission matrix G are essential for the conclusion (15).

If the medium is continuous everywhere and if ψ solves equation (1) in the whole region, then the amplitudes A and B will be constant throughout the medium, so that $G=I=$ identity matrix. In this case there will be no reflection. This occurs only in very special cases. One therefore concludes that a partition of the medium is almost always necessary and G is expressed as a product of transmission matrices corresponding to the partition points.

3. An upper bound to the reflectance

To evaluate an upper bound of the reflectance $|r|^2$ it is useful to introduce the matrix norm $\|G\|_\infty$ of transmission matrices. This norm is defined by the maximum absolute row sum (Isaacson and Keller 1966)

$$\|G\|_\infty = \max\{|G_{11}|+|G_{12}|, |G_{21}|+|G_{22}|\}.$$

According to equation (10) we have simply

$$\|G\|_\infty = \exp|\tau|. \tag{16}$$

The norm of a product of matrices is less than the product of the corresponding norms:

$$\|G\|_\infty \leqslant \prod_{j=1}^{j=n} \|G_j\|_\infty$$

so that

$$|\tau| \leqslant \sum_{j=1}^{j=n} |\tau_j|. \tag{17}$$

An upper bound for the reflectance follows,

$$|r|^2 = \tanh^2|\tau| \leqslant \tanh^2\left(\sum_{j=1}^{j=n} |\tau_j|\right). \tag{18}$$

As an example we consider step functions. In this case the result is extremely simple (I), $|\tau_j| = \frac{1}{2}|\Delta_j(\log k/\nu)|$, where $\Delta_j(\log k/\nu)$ is the jump of $\log k/\nu$ at the discontinuity. The inequality (18) results in

$$|r|^2 \leqslant \tanh^2[\tfrac{1}{2}\mathscr{V}(\log k/\nu)], \tag{19}$$

where $\mathscr{V}(\log k/\nu)$ designates the total variation of the step function $\log k/\nu$.

If there are n jumps up and down between two levels k_1/ν_1 and $(k_2/\nu_2) \geqslant (k_1/\nu_1)$, we have $\mathscr{V}(\log k/\nu) = n \log(k_2\nu_1/k_1\nu_2)$, so that

$$|r|^2 \leqslant \left[\frac{(k_2/\nu_2)^n - (k_1/\nu_1)^n}{(k_2/\nu_2)^n + (k_1/\nu_1)^n}\right]^2.$$

This upper limit can actually be reached if the thickness of each layer is 1/4 of the wavelength, a fact well known from the optics of thin films.

4. A condition for the eigenfrequencies of the field

We now wish to consider the determination of the eigenfrequencies of the field in a stratified medium. As an example we take the boundary conditions that the field u vanishes at the boundaries x_0 and x_1, i.e.

$$u(x_0) \propto \langle v_0, f_0\rangle = 0, \qquad u(x_1) \propto \langle v_0, G^\dagger f_1\rangle = 0,$$

where $G^\dagger$ is the adjoint of G, and $v_1 = Gv_0$. This system of equations has non-trivial solutions for $v_0 = (A_0, B_0)$ if its determinant vanishes, which means

$$\mathrm{Im}[\psi_1^*(\psi_0 G_{22} - G_{21}\psi_0^*)] = 0.$$

By analogy with the reflection coefficient r from (13) we define the transmission coefficient $d = T\psi_n(x_n)/\psi_0(x_1)$. If $r(x_0, x_1)$ and $d(x_0, x_1)$ refer to the medium between x_0 and x_1, the above condition can be written as

$$\mathrm{Im}\{[1 + r(x_0, x_1)]/d(x_0, x_1)\} = 0. \tag{20}$$

This exact result is very useful if r and d can be explicitly calculated by means of some approximation. Frequencies for which r and d satisfy the condition (20) are the eigenfrequencies of the field. For instance, the first WKB approximation gives $r = 0$ and

$$d = \exp\left(\mathrm{i} \int_{x_0}^{x_1} k(y)\,\mathrm{d}y\right),$$

so that the well known condition

$$\int_{x_0}^{x_1} k(y)\,\mathrm{d}y = m\pi, \qquad m = 1, 2, \ldots,$$

follows. An improved approximation for r and d can be found in (I).

5. Reflection due to a discontinuous derivative of the refractive index

Suppose that in the vicinity of the origin k is linear, $k(x) = a + bx$, with b_+ and b_- on the right and on the left respectively. Thus the wavenumber is continuous, but not its derivative. The task of this section is to determine the transmission matrix corresponding to such a discontinuity. Only the high frequency limit for waves of normal incidence ($\nu = 1$) will be considered.

First we look for exact solutions of equation (1) for a linearly varying k. The desired complex conjugate solutions can be expressed in terms of Hankel functions (Jahnke *et al* 1960):

$$\psi = \exp[-\mathrm{i}(a^2/2|b| - 3\pi/8)](a + bx)^{1/2} H^{(1)}_{1/4}[(a + bx)^2/2|b|]. \tag{21}$$

The Wronskian equals

$$W = -\mathrm{i}8b/\pi, \tag{22}$$

so that Im W is negative if b is positive. For negative b the function ψ must be replaced by its complex conjugate. The constant exponential factor in front is so arranged that for a finite x and in the limit $|b| \to 0$ the function $|W|^{-1/2}\psi$ approaches $(2a)^{-1/2}\exp(\mathrm{i}ax)$.

For sufficiently high frequencies we may use the asymptotic expression for $H^{(1)}_{1/4}$, so that equation (21) reduces to

$$\psi = \{4|b|/[\pi(a + bx)]\}^{1/2}\exp(\mathrm{i}ax), \tag{23}$$

independent of the sign of b. To this degree of approximation the Wronskian $W = -\mathrm{i}8|b|/\pi$ remains unchanged. Then calculation in the sense of equations (11) results in

$$G_{11} = 1 - \mathrm{i}\Delta b/4a^2, \qquad G_{12} = -\mathrm{i}\Delta b/4a^2, \tag{24}$$

where $\Delta b = b_+ - b_-$. A simple expression for r is obtained from equation (13), namely

$$r = -\mathrm{i}(\Delta b/4a^2)/(1 + \mathrm{i}(\Delta b/4a^2)). \tag{25}$$

On neglecting the second term in the denominator we rewrite this in terms of the wave number k and the jump $\Delta k'$ of its derivative,

$$r \simeq -\mathrm{i}\Delta k'/4k^2. \tag{26}$$

This result is in agreement with the expression given by Ginzburg (1961).

It is instructive to compare this result with that for a step of the wave number. In the latter case the reflection coefficient is independent of frequency, and the phase change of the reflected wave equals zero or π. On the other hand if only the derivative of k is discontinuous, r is inversely proportional to the frequency and the phase change equals $-\pi/2$ or $\pi/2$ for positive and negative $\Delta k'$, respectively.

6. Comments

The matrix formalism developed in § 2, which really seems elementary provides a useful algorithm in calculating reflectivity in cases where the exact solutions in each region of the medium are known, but must be related to each other at the boundary between the domains. The procedure simply replaces the usual treatment of boundary conditions. How it can work in specific cases will be shown in a separate work dealing with reflection from an asymmetric stratified medium with a resonance point. In the present paper my intention was to show that the algorithm developed so far enables us to derive some general relations, which are difficult to see if the boundary conditions are treated in a conventional manner. On the other hand, a more important application is expected through elaborating an improved explicit and analytic approximation for the reflectivity of a stratified medium. This would be a generalisation of the method in the previous paper I. A preliminary attempt based on piecewise linear functions for the wave number k shows some promise in this respect.

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